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Balanced Fiber Bundles and GKM Theory

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Let T be a torus and B a compact T-manifold. Goresky et al. show in [3] that if B is (what was subsequently called) a GKM manifold, then there exists a simple combinatorial description of the equivariant cohomology ring $H_T^*(B)$ as a subring of $H_T^*(B^T)$. In this paper, we prove an analog of this result for T-equivariant fiber bundles: we show that if M is a T-manifold and $\pi : M \to B$ a fiber bundle for which π intertwines the two T-actions, there is a simple combinatorial description of $H_T^*(M)$ as a subring of $H_T^*(\pi^{-1}(B^T))$. Using this result, we obtain fiber bundle analogs of results of Guillemin et al. [4] on GKM theory for homogeneous spaces.

1 Introduction

Let $T = (S^1)^n$ be an n-dimensional torus and M a compact, connected T-manifold. We recall that the equivariant cohomology $H_T^*(M) = H_T^*(M; \mathbb{R})$ of M is defined as the usual cohomology of the quotient $(M \times E)/T$, where E is the total space of the classifying bundle of the group T. Let

$$\pi : M \to B \tag{1.1}$$

be a T-equivariant fiber bundle. We will assume that the base B is simply connected and that the typical fiber is connected.

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Then one obtains a fiber bundle

$$(M \times E)/T \to (B \times E)/T \tag{1.2}$$

and a Serre–Leray spectral sequence relating the equivariant cohomology groups of M and B; the E_2-term of this spectral sequence is the product

$$H^*(F) \otimes H^*((B \times E)/T), \tag{1.3}$$

where F is the fiber of the bundle (1.2) and hence of the bundle (1.1). Thus, if the spectral sequence collapses at this stage, one obtains an isomorphism of additive cohomology

$$H_T^*(M) \simeq H^*(F) \otimes H_T^*(B). \tag{1.4}$$

However, this isomorphism does not say much about how the ring structure of $H_T^*(B)$ and $H_T^*(M)$ are related. One of the main goals of this paper is to address that question. We begin by recalling that one approach for computing the equivariant cohomology ring of a T-manifold M is by Kirwan localization. Suppose that $H_T^*(M)$ is free as a module over $H_T^*(pt) \simeq \mathbb{S}(\mathfrak{t}^*)$, the symmetric algebra of $\mathfrak{t}^*$. Then the restriction map

$$i^* : H_T^*(M) \to H_T^*(M^T)$$

is injective and hence computing $H_T^*(M)$ reduces to computing the image of $H_T^*(M)$ in $H_T^*(M^T)$. If M^T is finite, then

$$H_T^*(M^T) \simeq \bigoplus_{p \in M^T} \mathbb{S}(\mathfrak{t}^*).$$

Determining where $H_T^*(M)$ sits inside this sum is a challenging combinatorial problem. However, one class of spaces for which this problem has a simple and elegant solution is the one introduced by Goresky–Kottwitz–MacPherson in their seminal paper [3]. These are now known as *GKM spaces*.

Definition 1.1. A compact T-manifold M is a GKM manifold if:

(a) M^T is finite.

(b) M is equivariantly formal, that is,

$$H_T^*(M) \simeq H^*(M) \otimes \mathbb{S}(\mathfrak{t}^*)$$

as $\mathbb{S}(\mathfrak{t}^*)$-modules.

(c) For every codimension one subtorus $T' \subset T$, the connected components of $M^{T'}$ are either points or two-spheres. □

If S is one of the edge two-spheres, then S^T consists of exactly two T-fixed points, p and q (the "North" and "South" poles of S). To each GKM space M we attach a graph $\Gamma = \Gamma_M$ with set of vertices $V_\Gamma = M^T$, and edges corresponding to these two-spheres. If M has an invariant almost complex or symplectic structure, then the isotropy representations on tangent spaces at fixed points are complex representations and their weights are well-defined. In particular, condition (c) is equivalent to:

(c′) For every fixed point $p \in M^T$, the weights of the isotropy representation of T on T_pM are pairwise linearly independent.

These data determine a map

$$\alpha : E_\Gamma \to \mathbb{Z}_T^*$$

of oriented edges of Γ into the weight lattice of T. This map assigns to the edge (two-sphere) S, joining p to q and oriented from p to q, the weight of the isotropy representation of T on the tangent space to S at p. The map α is called the *axial function* of the graph Γ. We use it to define a subring $H_\alpha^*(\Gamma)$ of $H_T^*(M^T)$ as follows. Each element of $H_T^*(M^T)$ can be regarded as a function that assigns to each $p \in M^T$ an element $c(p)$ of $H_T^*(pt) = \mathbb{S}(\mathfrak{t}^*)$, hence $H_T^*(M^T)$ is identified with the ring $\mathrm{Maps}(V_\Gamma, \mathbb{S}(\mathfrak{t}^*))$.

Definition 1.2. The cohomology ring of (Γ, α), denoted by $H_\alpha^*(\Gamma)$, is the subring of $\mathrm{Maps}(V_\Gamma, \mathbb{S}(\mathfrak{t}^*))$ consisting of the maps c that have the property that for each edge e of Γ, with vertices p and q as endpoints, $c(p)$ and $c(q)$ have the same image in $\mathbb{S}(\mathfrak{t}^*)/\langle \alpha_e \rangle$. □

Note that without the invariant almost complex or symplectic structure, the isotropy representations are only real representations and the weights are defined only up to sign; however, that does not change the construction of $H_\alpha^*(\Gamma)$.

For GKM spaces, a direct consequence of a theorem of Chang and Skjelbred [2, Lemma 2.3] is that $H_\alpha^*(\Gamma)$ is the image of i^* (see [3, Theorem 1.2.2]), and therefore there is an isomorphism of rings

$$H_T^*(M) \simeq H_\alpha^*(\Gamma). \tag{1.5}$$

One of our main results is a generalization of (1.5) for T-equivariant fiber bundles

$$\pi : M \to B \tag{1.6}$$

for which the total space M is equivariantly formal and the base B is a GKM space. By the Kirwan Theorem, the composite map

$$H_T^*(M) \to H_T^*(\pi^{-1}(B^T)) \to H_T^*(M^T)$$

is injective. Hence one has an injective homomorphism of rings

$$H_T^*(M) \to \bigoplus_{p \in B^T} H_T^*(\pi^{-1}(p)), \tag{1.7}$$

and so to determine the ring structure of $H_T^*(M)$ it suffices to determine the image of this mapping. This we will do by a GKM type recipe similar to (1.5).

Let $(\Gamma = \Gamma_B, \alpha)$ be the GKM graph associated to B, and for $p \in B^T$ (i.e., a vertex of Γ) let $F_p = \pi^{-1}(p)$. If e is an edge of Γ joining the vertices p and q, and T_e is the subtorus of T with Lie algebra $\ker \alpha_e$, then F_p and F_q are isomorphic as T_e-spaces and hence

$$H_T^*(F_p)/\langle \alpha_e \rangle = H_T^*(F_q)/\langle \alpha_e \rangle. \tag{1.8}$$

We denote the ring (1.8) by $\mathcal{R}_e$ and we will prove the following generalization of (1.5).

Theorem 1.3. A function

$$c : V_\Gamma \to \bigoplus_{p \in B^T} H_T^*(F_p), \; c(p) \in H_T^*(F_p)$$

is in the image of (1.7) if and only if for every edge $e = (p, q)$ of Γ, the images of $c(p)$ and $c(q)$ in $\mathcal{R}_e$ coincide. □

One of our main applications of this result will be a fiber bundle version of the main result in [4]. In [4], it is shown that if G is a compact semisimple Lie group, T a Cartan subgroup, and K a closed subgroup of G, then the following conditions are equivalent:

(1) the action of T on G/K is GKM;
(2) the Euler characteristic of G/K is nonzero;
(3) K is of maximal rank, that is, $T \subset K$.

Moreover, for homogeneous spaces of the form G/K one has a description, due to Borel, of the equivariant cohomology ring of G/K as a tensor product

$$H_T^*(G/K) = \mathbb{S}(\mathfrak{t}^*)^{W_K} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_G}} \mathbb{S}(\mathfrak{t}^*) \tag{1.9}$$

and in [4] it is shown how to reconcile (1.9) with the description (1.5).

Suppose now that $M \to B$ is a T-equivariant fiber bundle with M equivariantly formal and B a GKM space. Let $p \in B^T$ be a fixed point for the T-action on B and $\pi_1(\Gamma, p)$ the group of paths $(e_1, e_2, \ldots, e_k)$ of oriented edges on Γ starting and ending at p. Modulo some hypotheses which we will spell out more carefully in Section 4, the group $\pi_1(\Gamma, p)$ induces an action of a group W_p (which we will call the *Weyl group of* p) both on $H_T^*(F_p)$ and on $\mathbb{S}(\mathfrak{t}^*)$. Fiber bundles satisfying those conditions will be called *balanced*. We will then show that there is a canonical embedding of $H_T^*(F_p)^{W_p}$ into $H_T^*(M)$ and its image generates $H_T^*(M)$ as a module over $H_T^*(B)$. One of the main results of this paper (see Theorem 4.2) is the following: if Γ_B is the GKM graph of B, then there is a canonical isomorphism of rings

$$H_T^*(M) \simeq H_T^*(F_p)^{W_p} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_p}} H_\alpha(\Gamma_B). \tag{1.10}$$

We apply this result to fiber bundles of the type $G/K_1 \to G/K$, with $K_1 \subset K$, and obtain a description of the cohomology ring of G/K_1 that will be of Borel type on the fibers and of GKM type on the base. This result will enable one to interpolate between two (in principle) very different descriptions of the ring $H_T^*(G/K)$, as we have already shown in special cases in [5].

A few words about the organization of this paper. In Section 2, we generalize the Chang–Skjelbred theorem to equivariant fiber bundles, and in Section 3 we use this result to prove Theorem 1.3. In Section 4, we construct the group W_p and prove the isomorphism (1.10). In Section 5, we describe in more detail the results of Guillemin et al. [4] alluded to above and show that the fiber bundles $G/K_1 \to G/K$ satisfy the conditions of Theorem 4.2, and in Section 6 we describe some connections between the results of this paper and results of Guillemin et al. [5], where we work out the implications of this theory in much greater detail for the classical flag varieties of type A_n, B_n, C_n, and D_n.

The results of Section 6 are also related to those of [9], in which the authors use symplectic techniques to compute the equivariant cohomology ring of a compact symplectic manifold with a Hamiltonian torus action that fibers equivariantly over another compact symplectic manifold with a Hamiltonian action of the same torus. In particular, the details of classical flag varieties are also analyzed.

The results of this paper are closely related to the results of [6**?**], the topic of which is K-theoretic aspects of GKM theory. In particular, in [6], we have shown how the graph theoretical analogs of the results of this paper can be extended to the K-theoretic setting, and in some work-in-progress we have also been able to show that there is K-theoretic version of the Chang–Skjelbred theorem of Section 2 and that it gives one an effective way of computing the K-groups of balanced fiber bundles.

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2 The Chang–Skjelbred Theorem for Fiber Bundles

Before proving the main theorem of this section, we recall the Chang–Skjelbred theorem [2, Lemma 2.3].

Let M be a compact manifold with a smooth action of a compact torus T, and assume that M is equivariantly formal. Let $K \subset T$ be an isotropy group of M, that is, there exists $x \in M$ such that $K = \{g \in T \mid g \cdot x = x\}$. From the compactness of M it follows that the family of groups which occur as isotropy groups of M is finite. Let $K_1, \ldots, K_N$ be the family of codimension one isotropy groups of M, and M^{K_j} the subset of points of M fixed by K_j, for every $j = 1, \ldots, N$. From the commutativity of the diagram given by inclusions of T-invariant spaces

$$\begin{tikzcd} M^T \arrow[rr, "i"] \arrow[dr, "i_{K_j}"'] & & M \\ & M^{K_j} \arrow[ur] & \end{tikzcd}$$

we obtain the following commutative diagram in equivariant cohomology:

$$\begin{tikzcd} H_T^*(M) \arrow[rr, "i^*"] \arrow[dr] & & H_T^*(M^T) \\ & H_T^*(M^{K_j}) \arrow[ur, "i^*_{K_j}"'] & \end{tikzcd}$$

which implies that

$$i^*(H_T^*(M)) \subseteq \bigcap_{j=1}^{N} i_{K_j}^*(H_T^*(M^{K_j})).$$

Lemma 2.3 in [2] asserts that the opposite inclusion holds as well.

Theorem 2.1 (Chang–Skjelbred)**.** Let M be a compact T-manifold, and assume that M is equivariantly formal. Let $K_1 \ldots, K_N$ be the subtori of codimension one which occur as isotropy groups of points of M. Then

$$i^*(H_T^*(M)) = \bigcap_{j=1}^{N} i_{K_j}^*(H_T^*(M^{K_j})). \tag{2.1}$$

□

Before proving the main theorem of this section we recall some standard facts about T-manifolds. (For a detailed exposition we refer to [7, Sections 11.3-4]; see also [1].)

Let A be a finitely generated $\mathbb{S}(\mathfrak{t}^*)$-module. The annihilator ideal I_A of A is

$$I_A = \{f \in \mathbb{S}(\mathfrak{t}^*),\ \ fA = 0\},$$

and the support of A is the algebraic variety in $\mathfrak{t} \otimes \mathbb{C}$ associated with this ideal, that is,

$$\text{supp } A = \{x \in \mathfrak{t} \otimes \mathbb{C},\ f(x) = 0 \text{ for all } f \in I_A\}.$$

Theorem 2.2 ([7, Theorem 11.4.1])**.** Let M be a compact T-manifold and X a closed T-invariant submanifold. Then the supports of the modules $H_T^*(M \setminus X)$ and $H_T^*(M \setminus X)_c$ are contained in the set

$$\bigcup_K \mathfrak{k} \otimes \mathbb{C} \tag{2.2}$$

the union being over the subgroups K which occur as isotropy groups of points of $M \setminus X$. □

Here, $H_T^*(\cdot)_c$ denotes the equivariant cohomology with compact supports. Moreover, the following sequence:

$$H_T^k(M \setminus X^T)_c \longrightarrow H_T^k(M) \longrightarrow H_T^k(X^T) \longrightarrow H_T^{k+1}(M \setminus X^T)_c \tag{2.3}$$

is exact.

Theorem 2.3 ([7, Theorem 11.4.2]). Let M and X be as in Theorem 2.2, and $i : X \to M$ the inclusion. The kernel and cokerel of the map $i^* : H_T^*(M) \to H_T^*(X)$ are supported in the set (2.2). □

Let $\pi : M \to B$ be a T-equivariant fiber bundle with M equivariantly formal and B a GKM space. Let K_j, $j = 1, \ldots, N$ be the codimension one isotropy groups of B and let $\mathfrak{k}_j$ be the Lie algebra of K_j.

Lemma 2.4. Let $x \in B$, $K = \{g \in T \mid g \cdot x = x\}$ its isotropy group, and $\mathfrak{k}$ the Lie algebra of K. Then

$$\mathfrak{k} = \bigcap_{r=1}^{m} \mathfrak{k}_{j_r}$$

for some multi-index $1 \leqslant j_1 < \cdots < j_m \leqslant N$. □

Proof. Let $Y \subset B^K \subset B$ be the connected component through x of the subspace fixed by K. The tangent space at x, T_xB, has a natural T action, and can be written as the direct sum of two T-invariant subspaces $T_xY \oplus N_x$, where T_xY denotes the tangent space of Y at x, and N is the normal bundle with respect to a T-invariant metric on B. On the other hand,

$$T_xB = \bigoplus_{j=1}^{n} \mathbb{L}_{\alpha_j}$$

for some $\alpha_1, \ldots, \alpha_n \in \mathfrak{t}^*$, where $\mathbb{L}_{\alpha_j}$ denotes the tangent space to the T-invariant sphere containing x, fixed by $\exp(\ker(\alpha_j)) \subset T$, for every $j = 1, \ldots, n$. Thus

$$T_xY = \bigoplus_{r=1}^{m} \mathbb{L}_{\alpha_{j_r}}$$

for some multi-index $1 \leqslant j_1 < \cdots < j_m \leqslant N$ and

$$\mathfrak{k} = \bigcap_{r=1}^{m} \ker(\alpha_{j_r}) = \bigcap_{r=1}^{m} \mathfrak{k}_{j_r}.$$ ■

For any subgroup K of T let $X^K = \pi^{-1}(B^K)$, where $B^K \subset B$ is the set of points in B fixed by K. Then X^K is a closed T-invariant submanifold of M. In particular,

$X^T = \pi^{-1}(B^T)$, the inverse image of the T-fixed points of B. From Lemma 2.4 and Theorem 2.2 one obtains the following.

Theorem 2.5. The $\mathbb{S}(\mathfrak{t}^*)$-module $H_T^*(M \setminus X^T)$ is supported on the set

$$\bigcup_{i=1}^{N} \mathfrak{k}_i \otimes \mathbb{C}. \tag{2.4}$$

□

Since $H_T^*(M)$ is a free $\mathbb{S}(\mathfrak{t}^*)$-module, by the exactness of (2.3) and Theorem 2.5 we have the following theorem.

Theorem 2.6. Let $i : X^T \to M$ be the inclusion, and $i^* : H_T(M) \to H_T(X^T)$ the induced map in equivariant cohomology. Then i^* is injective and coker(i^*) is supported on $\bigcup_{i=1}^{N} \mathfrak{k}_i \otimes \mathbb{C}$. □

As a consequence, we obtain the following corollary.

Corollary 2.7. If $e \in H_T^*(X^T)$, then there exist nonzero weights $\alpha_1, \ldots, \alpha_r$ such that $\alpha_j = 0$ on some $\mathfrak{k}_j$ and

$$\alpha_1 \cdots \alpha_r e \in i^*(H_T^*(M)). \tag{2.5}$$

□

The next theorem is a fiber bundle version of the Chang–Skjelbred theorem.

Theorem 2.8. Let $i : X^T \to M$ be the inclusion, and $i^* : H_T(M) \to H_T(X^T)$ the induced map in equivariant cohomology. Then the image of i^* is the ring

$$\bigcap_{j=1}^{N} i_{K_j}^* H_T^*(X^{K_j}), \tag{2.6}$$

where i_{K_j} denotes the inclusion of X^T into X^{K_j}. □

Proof. Via the inclusion i^* we can view $H_T^*(M)$ as a submodule of $H_T^*(X^T)$. Let $e_1, \ldots, e_m$ be a basis of $H_T^*(M)$ as a free module over $\mathbb{S}(\mathfrak{t}^*)$. Then by Corollary 2.7 for any $e \in H_T^*(X^T)$

we have

$$\alpha_1 \cdots \alpha_r e = \sum f_j e_j, \quad f_j \in \mathbb{S}(\mathfrak{t}^*).$$

Then $e = \sum \frac{f_j}{p} e_j$, where $p = \alpha_1 \cdots \alpha_r$. If f_j and p have a common factor we can eliminate it and write e uniquely as

$$e = \sum \frac{g_j}{p_j} e_j \tag{2.7}$$

with $g_j \in \mathbb{S}(\mathfrak{t}^*)$, p_j a product of a subset of the weights $\alpha_1, \ldots, \alpha_r$ and p_j and g_j relatively prime.

Suppose that K is an isotropy subgroup of B of codimension one and e is in the image of $H_T^*(X^K)$. By Theorem 2.3 the cokernel of the map $H_T^*(M) \to H_T^*(X^K)$ is supported on the subset $\cup \mathfrak{k}_j \otimes \mathbb{C}$, $\mathfrak{k}_j \neq \mathfrak{k}$ of (2.4), and hence there exists weights $\beta_1, \ldots, \beta_r$, β_j vanishing on some $\mathfrak{k}_j$ but not on $\mathfrak{k}$, such that

$$\beta_i \cdots \beta_r e = \sum f_j e_j.$$

Thus the p_j in (2.7), which is a product of a subset of the weights $\alpha_1, \ldots, \alpha_r$, is a product of a subset of weights none of which vanish on $\mathfrak{k}$. Repeating this argument for all the codimension one isotropy groups of B we conclude that the weights in this subset cannot vanish on any of these $\mathfrak{k}$'s, and hence is the empty set, that is, $p_j = 1$. Then if e is in the intersection (2.6), e is in $H_T^*(M)$. ■

Remark 2.9. Observe that Theorem 2.1 in the case in which M is GKM is a special case of Theorem 2.8, obtained for $M = B$ and π the identity. □

3 Fiber Bundles over GKM Spaces

We can now prove Theorem 1.3, the fiber bundle version of the GKM result.

We start with the case $B = \mathbb{C}P^1$. Let $\pi : M \to B = \mathbb{C}P^1$ be a T-equivariant fiber bundle with M equivariantly formal. If T acts on $\mathbb{C}P^1$ with weight α_e, then the codimension one subtorus $T_e = \exp(\ker \alpha_e) \subset T$ fixes the points of B, and the action of the quotient group T/T_e is effective. The fixed point set B^T consists of two points, p and q and its inverse image under π, $X^T = \pi^{-1}(B^T)$, consists of the two fibers $F_p = \pi^{-1}(p)$ and $F_q = \pi^{-1}(q)$. If $i : X^T \to M$ is the inclusion, then the map $i^* : H_T^*(M) \to H_T^*(X^T)$ is injective (see (1.7)). Recall that F_p and F_q are isomorphic as T_e-spaces and that (see (1.8))

$$H_T^*(F_p)/\langle \alpha_e \rangle = H_{T_e}^*(F_p) = H_{T_e}^*(F_q) = H_T^*(F_q)/\langle \alpha_e \rangle = \mathcal{R}_e.$$

Let r_p and r_q be the projections of $H_T^*(F_p)$ and $H_T^*(F_q)$, respectively, onto $\mathcal{R}_e$.

Theorem 3.1. The image of the map

$$H_T^*(M) \to H_T^*(X^T)$$

is the set of pairs $(f_p, f_q) \in H_T^*(F_p) \oplus H_T^*(F_q)$ satisfying $r_p(f_p) = r_q(f_q)$. □

Proof. Let S^1 be a circle complement of T_e in T, that is, $T = T_e \times S^1$. Then S^1 acts freely on $\mathbb{C}P^1 \setminus \{p, q\}$ and the quotient by S^1 of this action is the interval $(0, 1)$, so one has an isomorphism of T_e spaces

$$(M \setminus X^T)/S^1 = F \times (0, 1), \tag{3.1}$$

where, as T_e-spaces, $F = F_p = F_q$.

Consider now the long exact sequence (2.3). Since i^* is injective this becomes a short exact sequence

$$0 \to H_T^k(M) \to H_T^k(X^T) \to H_T^{k+1}(M \setminus X^T)_c \to 0. \tag{3.2}$$

Since S^1 acts freely on $M \setminus X^T$, we have

$$H_T^{k+1}(M \setminus X^T)_c = H_{T_e}^{k+1}((M \setminus X^T)/S^1)_c$$

and by fiber integration one obtains from (3.1)

$$H_{T_e}^{k+1}((M \setminus X^T)/S^1)_c = H_{T_e}^k(F).$$

Then the sequence (3.2) becomes

$$0 \to H_T^k(M) \to H_T^k(F_p) \oplus H_T^k(F_q) \to H_{T_e}^k(F) \to 0, \tag{3.3}$$

where the third arrow is the map

$$H_T^*(X^T) = H_T^*(F_p) \oplus H_T^*(F_q) \to H_{T_e}^*(F)$$

sending $f_p \oplus f_q$ to $-r_p(f_p) + r_q(f_q)$. (The $-r_p$ in the first term is due to the fact that the fiber integral

$$H_c^{k+1}(F \times (0,1)) \to H^k(F)$$

depends on the orientation of $(0, 1)$: the standard orientation for $F_q \times (0,1) \to F_q$ and the reverse orientation for $F_p \times (0,1) \to F_p$.) ■

Theorem 1.3 follows from Theorem 2.8 by applying Theorem 3.1 to all edges of the GKM graph of B.

4 Holonomy for Balanced Bundles

The goal of this section is to prove one of the main results of this paper, namely the fiber bundle analog of (1.9) we mentioned in the introduction (see (1.10)).

We begin by recalling the Cartan model for equivariant cohomology, considering the special case of torus actions. Let M be a smooth manifold and $\rho : T \times M \to M$ a smooth action of a torus T on M. Let $\mathfrak{t}$ be the Lie algebra of T, and for $\xi \in \mathfrak{t}$, let ξ_M be the corresponding vector field on M, given at $x \in M$ by

$$\xi_M(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(\exp(t\xi), x).$$

The space of cochains

$$\Omega_T(M) = \Omega(M)^T \otimes \mathbb{S}(\mathfrak{t}^*)$$

can be regarded as the space of T-invariant polynomial maps

$$p : \mathfrak{t} \to \Omega(M), \tag{4.1}$$

from $\mathfrak{t}$ to the space of differential forms $\Omega(M)$.

The coboundary operator $\mathrm{d}_T : \Omega_T(M) \to \Omega_T(M)$ is given by

$$(\mathrm{d}_T p)(\xi) = \mathrm{d}(p(\xi)) + \iota(\xi_M) p(\xi), \tag{4.2}$$

where d is the usual differential and ι is the interior product of a form by a vector field. Then $H_T^*(M)$ is the cohomology of the complex $(\Omega_T(M), \mathrm{d}_T)$.

If $\tau : T \to T$ is an automorphism of T, then we define a second action of T on M, $\rho^\tau : T \times M \to M$, $\rho^\tau(a, x) = \rho(\tau(a), x)$. Let $(\Omega_T(M)^\tau, \mathrm{d}_T^\tau)$ be the complex associated to ρ^τ and $H_T^*(M)^\tau$ the corresponding equivariant cohomology ring. The invariant forms are the same, $\Omega_T(M)^\tau = \Omega_T(M)$, but the coboundary operators are in general different:

$$(\mathrm{d}_T^\tau(p))(\xi) = \mathrm{d}(p(\xi)) + \iota(\xi_M^\tau) p(\xi) = \mathrm{d}(p(\xi)) + \iota((\tau_*(\xi))_M) p(\xi),$$

where $\tau_* : \mathfrak{t} \to \mathfrak{t}$ is the corresponding automorphism of the Lie algebra $\mathfrak{t}$.

The automorphism τ induces an isomorphism $\tau^* : \Omega_T(M) \to \Omega_T(M)$,

$$(\tau^*(p))(\xi) = p(\tau_*(\xi))$$

and the isomorphism τ^* commutes with the coboundary operators,

$$\tau^* \mathrm{d}_T = \mathrm{d}_T^\tau \tau^*, \tag{4.3}$$

and therefore it induces an isomorphism $\tau^* : H_T^*(M) \to H_T^*(M)^\tau$.

Suppose now that M and N are T-manifolds and $f : M \to N$ is a smooth map that intertwines the T-action on M with the τ-twisted action on N,

$$f(a \cdot x) = \tau(a) \cdot f(x),$$

for all $a \in T$ and $x \in M$. The pull-back map $f^* : \Omega(N) \to \Omega(M)$ satisfies

$$\rho_{M,a}^* \circ f^* = f^* \circ \rho_{N,\tau(a)}^*,$$

and extends to a map $f^* : \Omega_T(N) \to \Omega_T(M)$ by $(f^* p)(\xi) = f^*(p(\xi))$, and this map intertwines the τ^* operations for M and N,

$$\tau^* f^* = f^* \tau^*. \tag{4.4}$$

Moreover, $\iota(\xi_M) f^* = f^* \iota(\tau_*(\xi)_N)$ implies that

$$\mathrm{d}_T f^* = f^* \mathrm{d}_T^\tau, \tag{4.5}$$

where d_T is on M and d_T^τ on N.

Thus by (4.3) and (4.5) $\tau^* f^*$ intertwines the d_T operators on $\Omega_T(N)$ and $\Omega_T(M)$ and hence defines an morphism on cohomology

$$\tau^\# f^\# : H_T^*(N) \to H_T^*(M). \tag{4.6}$$

If the map $f : M \to N$ is a diffeomorphism, then $\tau^\# f^\#$ is an isomorphism.

Another property of $f^\#$ which we will need if the following. If T_e is a closed subgroup of T, with Lie algebra $\mathfrak{t}_e$, then one has restriction maps

$$\Omega_T \to \Omega_{T_e}, \quad p \to p|_{\mathfrak{t}_e},$$

and these induce maps in cohomology. If $\tau|_{T_e}$ is the identity, then the diagram

$$\begin{tikzcd} H_T^*(N) \arrow[r, "\tau^\# f^\#"] \arrow[d] & H_T^*(M) \arrow[d] \\ H_{T_e}^*(N) \arrow[r, "f^\#"] & H_{T_e}^*(M) \end{tikzcd} \tag{4.7}$$

commutes.

We will apply these observations to the fibers of the T-equivariant fiber bundle $M \to B$, and we begin by recalling a few elementary facts about holonomy. By equipping M with a T-invariant Riemannian metric, we obtain for each $m \in M$ an orthonormal complement in $T_m M$ of the tangent space at m to the fiber of π, that is, an "Ehresman connection". Thus, if p and q are points of B and γ is a curve joining p to q we obtain, by parallel transport, a diffeomorphism $f_\gamma : F_p \to F_q$, where $f_\gamma(m)$ is the terminal point of the horizontal curve in M projecting onto γ and having m as its initial point. Moreover, if γ and γ' are homotopic curves joining p to q, then the diffeomorphisms f_γ and $f_{\gamma'}$ are isotopic, that is, the isotopy class of f_γ depends only on the homotopy class of γ.

Suppose now that the base B is GKM, p and q are adjacent vertices of Γ_B, e is the edge joining them and S the two-sphere corresponding to this edge. We can then choose γ to be a longitudinal line on S joining the South pole p to the North pole q; since this line is unique up to homotopy, we obtain an intrinsically defined isotopy class of diffeomorphisms of F_p onto F_q. Moreover, since the Ehresman connection on M is T-invariant and T_e fixes γ, the maps in this isotopy class are T_e-invariant.

Definition 4.1. Let $M \to B$ be a T-equivariant fiber bundle over a GKM space B. Let $e = (p, q)$ be an edge of the GKM graph Γ_B. The T-equivariant fiber bundle $M \to B$ is *balanced at e* if there exists a diffeomorphism f_e in the isotopy class defined above and an automorphism τ_e of T such that f_e intertwines the T-action on F_p with the τ_e-twisted action of T on F_q:

$$f_e(ax) = \tau_e(a)\, f_e(x)$$

for all $a \in \tau_e$ and $x \in F_p$. The fiber bundle $M \to B$ is *balanced* if it is balanced at all edges of Γ_B. $\square$

It is clear that this τ_e, if it exists, is unique and restricts to the identity on T_e.

Suppose now that the T-equivariant bundle $M \to B$ is balanced. Let $\mathrm{Aut}(F_p)$ be the group of isotopy classes of diffeomorphisms of F_p, $\mathrm{Aut}(T)$ the group of automorphisms of T, and $\pi_1(\Gamma, p)$ the group of paths $\gamma = (e_1, e_2, \ldots, e_k)$ of oriented edges in Γ starting and ending at p. Given a path γ in $\pi_1(\Gamma, p)$ we have for each j a ring isomorphism

$$\tau_{e_j}^{\#} f_{e_j}^{\#} : H_T^*(F_{p_{j+1}}) \to H_T^*(F_{p_j}), \tag{4.8}$$

p_j and p_{j+1} being the initial and terminal vertices of e_j, and by composing these maps we obtain a ring automorphism, $\tau_{e_k}^{\#} f_{e_k}^{\#} \circ \cdots \circ \tau_{e_1}^{\#} f_{e_1}^{\#}$, of $H_T^*(F_p)$. Moreover, by (4.4) we can rewrite the factors in this product as $\tau_\gamma^{\#} f_\gamma^{\#}$ where $\tau_\gamma = \tau_{e_1} \circ \cdots \circ \tau_{e_k}$ and $f_\gamma = f_{e_1} \circ \cdots \circ f_{e_k}$. Thus the map $\gamma \to \tau_\gamma^{\#} f_\gamma^{\#}$ gives one a *holonomy* action of $\pi_1(\Gamma, p)$ on $H_T^*(F_p)$. We will denote the image of this map by W_p and call it the *Weyl group of p*. As a subgroup of $\mathrm{Aut}(F_p) \times \mathrm{Aut}(T)$, W_p acts on $H_T^*(F_p)$ and on $\mathbb{S}(\mathfrak{t}^*)$. The main result of this section is the following theorem.

Theorem 4.2. Let $\pi : M \to B$ be a balanced T-equivariant fiber bundle, such that the base B is a simply connected GKM space and the total space M is equivariantly formal. Moreover, suppose that the fiber F_p over a fixed point $p \in B^T$ is also equivariantly formal as a T-space. Then

$$H_T^*(M) \simeq H_T^*(F_p)^{W_p} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_p}} H_\alpha(\Gamma_B). \qquad \square$$

Proof. By (1.7), $H_T^*(M)$ is isomorphic to the subring of

$$\mathrm{Maps}_0 = \left\{ \phi : B^T \to \bigoplus_{q \in B^T} H_T^*(F_q) | \phi(q) \in H_T^*(F_q) \text{ for all } q \right\}$$

given by the conditions of Theorem 1.3. We identify $H_T^*(M)$ with that subring and consider $H_T^*(M) \subset \mathrm{Maps}_0$.

Let $c_p \in H_T^*(F_p)^{W_p}$. For $q \in B^T$, let $\gamma_{p,q}$ be a path in Γ joining p to q, and let $c_q \in H_T^*(F_q)$ be the class obtained by transporting c_p along $\gamma_{p,q}$. Because c_p is W_p-invariant, this class c_q is independent of the chosen path $\gamma_{p,q}$. Consider

$$\tilde{c} : B^T \to \bigoplus_{q \in B^T} H_T^*(F_q), \quad \tilde{c}(q) = c_q.$$

If q_1 and q_2 are adjacent vertices in Γ_B and e is the edge joining q_1 to q_2, then $c_{q_1} = \tau_e^* f_e^* c_{q_2}$ and hence by the commutative diagram (4.7), the images of c_{q_1} and c_{q_2} in the quotient space

$$H_T^*(F_{q_1})/\langle \alpha_e \rangle = H_T^*(F_{q_2})/\langle \alpha_e \rangle$$

are the same. Using Theorem 1.3, we conclude that $\tilde{c} \in H_T^*(M)$. We have therefore defined an injective morphism of rings

$$H_T^*(F_p)^{W_p} \to H_T^*(M),$$

and this map is also a morphism of $\mathbb{S}(\mathfrak{t}^*)^{W_p}$-modules. Tensoring with the pull-back

$$H_\alpha(\Gamma_B) \simeq H_T^*(B) \xrightarrow{\pi^*} H_T^*(M),$$

we obtain a morphism of rings

$$\Phi : H_T^*(F_p)^{W_p} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_p}} H_\alpha(\Gamma_B) \to H_T^*(M),$$

which acts on decomposable elements $c_p \otimes \omega$ by

$$\Phi(c_p \otimes \omega)(q) = \omega(q) c_q.$$

We will show that Φ is an isomorphism.

We first show that Φ is injective. Consider the double complex $(\Omega_T(F_p), d, \delta)$, with $(dh)(\xi) = d(h(\xi))$ and $(\delta h)(\xi) = \iota_{\xi^\#} h(\xi)$. Let $\gamma \in \pi_1(\Gamma, p)$ be a loop in Γ starting and ending at p and $f_\gamma : F_p \to F_p$ and $\tau_\gamma : T \to T$ the maps obtained by composing the corresponding transition maps along the edges of γ. The action of γ on $\Omega_T(F_p)$, given by

$$(\gamma \cdot h)(\xi) = f_\gamma^* h((\tau_\gamma)_*(\xi)),$$

preserves the bigrading of the double complex and commutes with the coboundary operators d and δ. Since F_p is by assumption equivariantly formal,

$$H_T^*(F_p) \simeq H^*(F_p) \otimes \mathbb{S}(\mathfrak{t}^*) \tag{4.9}$$

as $\mathbb{S}(\mathfrak{t}^*)$-modules. Moreover, since the W_p-action on $\Omega_T(F_p)$ commutes with the coboundary operators d and δ, the isomorphism (4.9) is W_p-equivariant. Since B is simply connected, the diffeomorphisms $f_\gamma : F_p \to F_p$ are (nonequivariantly) isotopic to the identity, so they act trivially on $H^*(F_p)$, and therefore

$$H_T^*(F_p)^{W_p} \simeq H^*(F_p) \otimes \mathbb{S}(\mathfrak{t}^*)^{W_p}.$$

Hence a basis $\tilde{c}_{1,p}, \ldots, \tilde{c}_{N,p}$ of $H^*(F_p)$ generates a basis $c_{1,p}, \ldots, c_{N,p}$ of $H_T^*(F_p)^{W_p}$ as a free module over $\mathbb{S}(\mathfrak{t}^*)^{W_p}$.

Every element $\eta \in H_T^*(F_p)^{W_p} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_p}} H_\alpha(\Gamma_B)$ is of the form

$$\eta = \sum_{k=1}^{N} c_{k,p} \otimes \omega_k$$

for some classes $\omega_k \in H_\alpha(\Gamma_B)$. If $\eta \in \ker \Phi$, then for every $q \in B^T$,

$$\sum_{k=1}^{N} \omega_k(q) c_{k,q} = \Phi(\eta)(q) = 0.$$

But for every q the classes $\{c_{1,q}, \ldots, c_{N,q}\}$ are a free set of generators for the $\mathbb{S}(\mathfrak{t}^*)$-module $H_T^*(F_q)$, and that implies that $\omega_1 = \cdots = \omega_N = 0$, hence $\eta = 0$. This proves that Φ is injective.

To prove the surjectivity, we keep track of bi-degrees. Consider the composition

$$\Psi : H^*(F_p) \otimes H_T^*(B) \to H_T^*(F_p)^{W_p} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_p}} H_\alpha(\Gamma_B) \xrightarrow{\Phi} H_T^*(M);$$

the first arrow is the linear map that sends $\tilde{c}_{j,p} \otimes \omega$ to $c_{j,p} \otimes \omega$ for all $j = 1, \ldots, N$. Therefore, $\Psi(\tilde{c}_{j,p} \otimes \omega)(q) = c_{j,p}(q)\omega(q)$ for all $q \in V_\Gamma$, for all $j = 1, \ldots, N$.

By assumption, the Serre–Leray spectral sequence associated with the fiber bundle π collapses at its E_2 stage. The E_2 term of this sequence is $H^*(F_p) \otimes H_T^*(B)$ and the E_∞ term is $H_T^*(M)$, hence

$$\bigoplus_{j+k=i} H^j(F_p) \otimes H_\alpha^k(\Gamma_B) = H_T^i(M).$$

Since Ψ preserves the grading and on each homogeneous component is an injective linear map between spaces of the same dimension, we conclude that Ψ is surjective, and therefore Φ is surjective as well. ∎

Apropos of the assumption that the F_p's are equivariantly formal as T spaces, we note:

(i) Since the fiber bundle, $M \to B$, is balanced, it suffices to assume this just for the "base" fiber, F_{p_0}, above a single $p_0 \in B^T$.

(ii) If M is a Hamiltonian T-space, then the fibers over points in B^T are Hamiltonian T-spaces as well, hence are equivariantly formal. (Note that in particular if M is a Hamiltonian GKM space, then the fibers over points in B^T are Hamiltonian GKM spaces.)

5 Homogeneous Fiber Bundles

We apply the results of the previous section to homogeneous fiber bundles, and we start by recalling some of the notation and results of Guillemin et al. [4].

Let G be a compact connected semisimple Lie group, T its Cartan subgroup, and K a closed subgroup of G containing T. Then, as asserted above, G/K is a GKM space. The proof of this consists essentially of describing explicitly the GKM structure of G/K in terms of the Weyl groups of G and K. We first note that for $K = T$, that is, for the generalized flag variety $M = G/T$, the fixed point set, M^T, is just the orbit of $N(T)$, the normalizer of T in G, through the identity coset, p_0, and hence M^T can be identified with $N(T)/T = W_G$. Let Δ_G be the set of roots, and Δ_G^+ a choice of positive roots. To show that M is GKM it suffices to check the GKM condition p_0. To do so, we identify the tangent space $T_{p_0}M$ with $\mathfrak{g}/\mathfrak{t}$ and identify $\mathfrak{g}/\mathfrak{t}$ with the sum of the positive root spaces

$$\mathfrak{g}/\mathfrak{t} = \bigoplus_{\alpha\in\Delta_G^+} \mathfrak{g}_\alpha, \tag{5.1}$$

the α' s being the weights of the isotropy representation of T on $\mathfrak{g}/\mathfrak{t}$. It then follows from a standard theorem in Lie theory that the weights are pairwise independent and this in turn implies that condition (c′) is satisfied at p_0.

To see what the edges of the GKM graph are at p_0 let $\chi_\alpha : T \to S^1$ be the character homomorphism

$$\chi_\alpha(t) = \exp \mathrm{i}\alpha(t),$$

let H_α be its kernel, and G_α the semisimple component of the centralizer $C(H_\alpha)$ of H_α in G. Then G_α is either SU(2) or SO(3), and in either case $G_\alpha p_0 \simeq \mathbb{C}P^1$. However since G_α centralizes H_α, $G_\alpha p_0$ is H_α-fixed and hence is the connected component of M^{H_α} containing p_0. Thus, the oriented edges of the GKM graph of M with initial point p_0 can be identified with the elements of Δ_G^+ and the axial function becomes the function which labels by α the oriented edge $G_\alpha p_0$. Moreover, under the identification $M^T = W_G$, the vertices that are joined to p_0 by these edges are of the form $\sigma_\alpha p_0$, where $\sigma_\alpha \in W_G$ is the reflection which leaves fixed the hyperplane ker $\alpha \subseteq \mathfrak{t}$ and maps α to $-\alpha$.

Letting $a \in N(T)$ and letting $p = ap_0$ one obtains essentially the same description of the GKM graph at p. Namely, denoting this graph by Γ, the following are true:

(1) the maps, $a \in N(T) \to ap_0$ and $a \in N(T) \to w \in N(T)/T$, set up a 1-to-1 correspondence between the vertices, M^T, of Γ and the elements of W_G;
(2) two vertices, w and w', are on a common edge if and only if $w' = w\sigma_\alpha$ for some $\alpha \in \Delta_G^+$;
(3) the edges of Γ containing $p = ap_0$ are in one-one correspondence with elements of Δ_G^+;
(4) for $\alpha \in \Delta_G^+$ the stabilizer group of the edge corresponding to α is $aH_\alpha a^{-1}$.

Via the fiber bundle $G/T \to G/K$ one obtains essentially the same picture for G/K. Namely let $\Delta_{G,K}^+ = \Delta_G^+ \setminus \Delta_K^+$.

Theorem 5.1 ([4, Theorem 2.4]). The homogeneous space G/K is a GKM space with GKM graph Γ, where

(1) the vertices of Γ are in 1-to-1 correspondence with the elements of W_G/W_K;
(2) two vertices $[w]$ and $[w']$ are on a common edge if and only if $[w'] = [w\sigma_\alpha]$ for some $\alpha \in \Delta^+_{G,K}$;
(3) the edges of Γ containing the vertex $[w]$ are in 1-to-1 correspondence with the roots in $\Delta^+_{G,K}$;
(4) if α is such a root the the stabilizer group of the $\mathbb{C}P^1$ corresponding to the edge is $aH_\alpha a^{-1}$ where a is a preimage in $N(T)$ of $w \in W_G$. □

Remark 5.2. The GKM graph that we have just described is not simple in general, that is, it will in general have more than one edge joining two adjacent vertices. There is, however, a simple sufficient condition for simplicity. □

Theorem 5.3 ([4]). If K is a stabilizer group of an element of $\mathfrak{t}^*$, that is, if G/K is a coadjoint orbit, then the graph we have constructed above is simple. □

Proposition 5.4. Let K_1 be a closed subgroup of K. Then the fiber bundle

$$G/K_1 \to G/K \tag{5.2}$$

is balanced. □

Proof. To show that this is balanced it suffices to show that it is balanced at the edges going out of the identity coset, p_0. However, if e is the edge joining p_0 to $\sigma_\alpha p_0$ and a is the preimage of σ_α in $N(T)$ then conjugation by a maps the fiber, $F_{p_0} = K/K_0$ of (5.2) at p_0 onto the fiber $F_p := aK/aK_0$ of (5.2) at $p = \sigma_\alpha p_0$, and conjugates the action of Γ on F_{p_0} to the twisted action, aTa^{-1}, of T on F_p. Moreover, since a is in the centralizer of H_α, this twisted action, restricted to H_α, coincides with the given action of H_α, that is, if $T_e = H_\alpha$, conjugation by a is a T_e-equivariant isomorphism of F_{p_0} onto F_p. Hence the fiber bundle (5.2) is balanced. ■

For the homogeneous fiber bundles described in this section, the assumption that the fibers F_p are equivariantly formal as T-spaces is equivalent to the assumption that the F_p's are GKM. To see this, we note that if G/K is equivariantly formal then $(G/K)^T$ has to be nonempty by Guillemin and Sternberg [7, Theorem 11.4.5] and hence for some $g \in G$, $g^{-1}Tg \subseteq K$. In other words K is of maximal rank and hence by the theorem in [4] that we cited above G/K is GKM.

6 Examples

The results of this paper are closely related to the combinatorial results of our recent article [5]. More explicitly, in [5] we develop a GKM theory for fiber bundles in which the objects involved: the base, the fiber and the total space of the fiber bundle, are GKM graphs. We then formulate, in this context, a combinatorial notion of "balanced", show that one has an analog of the isomorphism (1.11) and use this fact to define some new combinatorial invariants for the classical flag varieties of type A_n, B_n, C_n, and D_n. In this section, we will give a brief account of how these invariants can be defined geometrically by means of the techniques developed above.

Example 1. Let $G = \mathrm{SU}(n+1)$, $K = T$, the Cartan subgroup of diagonal matrices in $\mathrm{SU}(n+1)$, and $K_1 = S(S^1 \times U(n))$. Then $G/K_1 \simeq \mathbb{C}P^n$, the complex projective space. The Weyl group of $\mathrm{SU}(n+1)$ is isomorphic to S_{n+1}, the group of permutations on $n+1$ elements. Let $\mathcal{A}_n = G/K$ be the generic coadjoint orbit of type A_n; then $\mathcal{A}_n \simeq \mathcal{F}l(\mathbb{C}^{n+1})$, the variety of complete flags in $\mathbb{C}^{n+1}$. The fiber bundle

$$\pi : \mathcal{A}_n \simeq G/K \to G/K_1 \simeq \mathbb{C}P^n$$

sends a flag $V_\bullet$ to its one-dimensional subspace V_1. The fiber over a line L in $\mathbb{C}P^n$ is $\mathcal{F}l(\mathbb{C}^{n+1}/L) \simeq \mathcal{F}l(\mathbb{C}^n) \simeq \mathrm{SU}(n)/T'$, where T' is the Cartan subgroup of diagonal matrices in $\mathrm{SU}(n)$. The fibers inherit a T-action from $\mathcal{F}l(\mathbb{C}^{n+1})$, but are not T-equivariantly isomorphic. If p and q are fixed points for the T-action on $\mathbb{C}P^n$, then the fibers F_p and F_q are T_e-equivariantly isomorphic, where T_e is the subtorus fixing the $\mathbb{C}P^1$ in $\mathbb{C}P^n$ with poles p and q. The Weyl group W_p of the fiber at p is isomorphic to S_n, the Weyl group of $\mathrm{SU}(n)$, and the holonomy action of W_p on the equivariant cohomology of the fiber is equivalent to the induced action of S_n on the equivariant cohomology of the flag variety $\mathcal{F}l(\mathbb{C}^n)$.

We can iterate this fiber bundle and construct a tower of fiber bundles

$$\begin{tikzcd}
\mathrm{pt} \arrow[r] & \mathcal{A}_1 \arrow[r] \arrow[d] & \mathcal{A}_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{A}_{n-1} \arrow[r] \arrow[d] & \mathcal{A}_n \arrow[d] \\
 & \mathbb{C}P^1 & \mathbb{C}P^2 & & \mathbb{C}P^{n-1} & \mathbb{C}P^n
\end{tikzcd}$$

Using this tower, we construct a basis of invariant classes on $\mathcal{A}_n$ by repeatedly applying the isomorphism (1.10). A typical stage in the process is the following. By (1.10), we have

$$H_T(\mathcal{A}_k) = H_T(\mathcal{A}_{k-1})^{S_k} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{S_k}} H_T(\mathbb{C}P^k).$$

Suppose we have constructed a basis of invariant classes on $\mathcal{A}_{k-1}$; this is trivial for $\mathcal{A}_0 =$pt. We use, as a basis for $H_T(\mathbb{C}P^k)$, classes represented by powers of the equivariant symplectic form $\omega - \tau \in \Omega^2_T(\mathbb{C}P^k)$. The pull-backs of these classes to $\mathcal{A}_k$ are invariant under the holonomy action, and the classes given by the isomorphism (1.10) form a basis of the equivariant cohomology of $\mathcal{A}_k$. As shown in [5], this basis consists of classes that are invariant under the corresponding holonomy action. By iterating this process, we obtain an $\mathbb{S}(\mathfrak{t}^*)$-basis of $H_T(\mathcal{A}_n)$ consisting of S_{n+1}-invariant classes. The combinatorial version of this construction is given in [5, Section 5.1]. $\square$

Example 2. Let $G = \mathrm{SO}(2n+1)$, $K = T$ a maximal torus in G, and $K_1=\mathrm{SO}(2) \times \mathrm{SO}(2n-1)$. Then $G/K_1 \simeq \mathrm{Gr}_2^+(\mathbb{R}^{2n+1})$, the Grassmannian of oriented two planes in $\mathbb{R}^{2n+1}$. Let W_n be the Weyl group of $\mathrm{SO}(2n+1)$, that is, the group of signed permutations on n elements. Let $\mathcal{B}_n = G/K$ be the generic coadjoint orbit of type B_n and

$$\pi : \mathcal{B}_n \simeq G/K \to G/K_1 \simeq \mathrm{Gr}_2^+(\mathbb{R}^{2n+1})$$

the natural projection. Note that the fibers are isomorphic to $\mathcal{B}_{n-1}$, but not isomorphic as T-spaces, since the T-action on the pre-image of the T-fixed points of $\mathrm{Gr}_2^+(\mathbb{R}^{2n+1})$ changes. Moreover, for every T-fixed point $p \in \mathrm{Gr}_2^+(\mathbb{R}^{2n+1})$, the Weyl group W_p of the fiber F_p is isomorphic to W_{n-1}. Just as in the previous example, we have the following tower of fiber bundles:

$$\begin{array}{ccccccccc}
\mathrm{pt} & \longrightarrow & \mathbb{C}P^1 & \longrightarrow & \mathcal{B}_2 & \longrightarrow \cdots \longrightarrow & \mathcal{B}_{n-1} & \longrightarrow & \mathcal{B}_n \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{C}P^1 & & \mathrm{Gr}_2^+(\mathbb{R}^5) & & \mathrm{Gr}_2^+(\mathbb{R}^{2n-1}) & & \mathrm{Gr}_2^+(\mathbb{R}^{2n+1})
\end{array}$$

and we can use this tower to produce a basis of invariant classes on $\mathcal{B}_n$. Namely, by (1.10), we have

$$H_T(\mathcal{B}_k) = H_T(\mathcal{B}_{k-1})^{W_{k-1}} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_{k-1}}} H_T(\mathrm{Gr}_2^+(\mathbb{R}^{2k+1})).$$

Since the classes represented by powers of the equivariant symplectic form $\omega - \tau \in \Omega^2_T(\mathrm{Gr}_2^+(\mathbb{R}^{2k+1}))$ are a W_k-invariant basis for $H_T(\mathrm{Gr}_2^+(\mathbb{R}^{2k+1}))$, their pull-backs are classes on $\mathcal{B}_k$ invariant under the holonomy action. Thus, we can repeat the same argument of the previous example and produce a basis for $H_T(\mathcal{B}_n)$ consisting of W_n-invariant classes. $\square$

Example 3. Let $G = \mathrm{Sp}(n)$ be the symplectic group, $K = T$ a maximal torus in G, and $K_1 = S^1 \times \mathrm{Sp}(n-1)$. Then $G/K_1 \simeq \mathbb{C}P^{2n-1}$. Let W_n be the Weyl group of $\mathrm{Sp}(n)$, that is, the group of signed permutations on n elements. Let $\mathcal{C}_n = G/K$ be the generic coadjoint orbit of type C_n and

$$\pi : \mathcal{C}_n \simeq G/K \to G/K_1 \simeq \mathbb{C}P^{2n-1}$$

the natural projection. For every T-fixed point $p \in \mathbb{C}P^{2n-1}$, the Weyl group acting on the fiber F_p is isomorphic to W_{n-1}. Since the fibers are isomorphic to $\mathcal{C}_{n-1}$, we obtain the following tower of fiber bundles:

$$\begin{array}{ccccccccc}
\mathrm{pt} \longrightarrow & \mathbb{C}P^1 & \longrightarrow & \mathcal{C}_2 & \longrightarrow \cdots \longrightarrow & \mathcal{C}_{n-1} & \longrightarrow & \mathcal{C}_n \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& \mathbb{C}P^1 & & \mathbb{C}P^3 & & \mathbb{C}P^{2n-3} & & \mathbb{C}P^{2n-1}
\end{array}$$

By (1.10), we have that

$$H_T(\mathcal{C}_k) = H_T(\mathcal{C}_{k-1})^{W_{k-1}} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_{k-1}}} H_T(\mathbb{C}P^{2k-1}).$$

By taking classes represented by powers of the equivariant symplectic form $\omega - \tau \in \Omega^2_T(\mathbb{C}P^{2k-1})$ we obtain a W_k-invariant basis of $H_T(\mathbb{C}P^{2k-1})$, and iterating the same procedure as before, a W_n-invariant basis of $H_T(\mathcal{C}_n)$. $\square$

Example 4. Let $G = \mathrm{SO}(2n)$, $K = T$ a maximal torus in G, and $K_1 = \mathrm{SO}(2) \times \mathrm{SO}(2n-2)$. Then $G/K_1 \simeq \mathrm{Gr}_2^+(\mathbb{R}^{2n})$, the Grassmannian of oriented two planes in $\mathbb{R}^{2n}$. Let W_n be the Weyl group of $\mathrm{SO}(2n)$, that is, the subgroup of the signed permutation group of n elements, consisting of an even number of sign changes. Let $\mathcal{D}_n = G/K$ be the generic coadjoint orbit of type D_n and

$$\pi : \mathcal{D}_n \simeq G/K \to G/K_1 \simeq \mathrm{Gr}_2^+(\mathbb{R}^{2n})$$

the natural projection. For every T-fixed point $p \in \mathrm{Gr}_2^+(\mathbb{R}^{2n})$, the Weyl group acting on the fiber F_p is isomorphic to W_{n-1}. Since the fibers are isomorphic to $\mathcal{D}_{n-1}$, we obtain the following tower of fiber bundles:

$$\begin{array}{ccccccccc} \mathrm{pt} & \longrightarrow & \mathbb{C}P^1 \times \mathbb{C}P^1 & \longrightarrow & \mathcal{D}_3 & \longrightarrow \cdots \longrightarrow & \mathcal{D}_{n-1} & \longrightarrow & \mathcal{D}_n \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{C}P^1 \times \mathbb{C}P^1 & & \mathrm{Gr}_2^+(\mathbb{R}^6) & & \mathrm{Gr}_2^+(\mathbb{R}^{2n-2}) & & \mathrm{Gr}_2^+(\mathbb{R}^{2n}) \end{array}$$

By (1.10), we have that

$$H_T(\mathcal{D}_k) = H_T(\mathcal{D}_{k-1})^{W_{k-1}} \otimes_{\mathbb{S}(\mathfrak{t}^*)^{W_{k-1}}} H_T(\mathrm{Gr}_2^+(\mathbb{R}^{2k})).$$

By repeating an argument similar to the other examples, in order to produce a W_n-invariant basis of $H_T(\mathcal{D}_n)$, it is sufficient to exhibit a W_k-invariant basis of $H_T(\mathrm{Gr}_2^+(\mathbb{R}^{2k}))$ for every $k \geqslant 2$. To do this, let $x_1 - x_2, \ldots, x_{k-1} - x_k, x_{k-1} + x_k$ be a choice of simple roots of type D_k. Then the classes represented by powers of the equivariant symplectic form $\omega - \tau \in \Omega_T^2(\mathrm{Gr}_2^+(\mathbb{R}^{2n}))$ together with the class $\eta \in H_T^{2n-2}(\mathrm{Gr}_2^+(\mathbb{R}^{2n}))$ whose restriction at the fixed points is given by $x_1 \cdots x_k \tau^{-1}$, form a W_k-invariant basis of $H_T(\mathrm{Gr}_2^+(\mathbb{R}^{2n}))$. □

In [5], we also show how these iterated invariant classes relate to a better known family of classes generating the equivariant cohomology of flag varieties, namely the equivariant Schubert classes.

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